On a new proof of the Prime Number Theorem

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Abstract

A new elementary proof of the prime number theorem presented recently in the framework of a scale invariant extension of the ordinary analysis is re-examined and clarified further. Both the formalism and proof are presented in a much more simplified manner. Basic properties of some key concepts such as infinitesimals, the associated nonarchimedean absolute values, invariance of measure and cardinality of a compact subset of the real line under an IFS are discussed more thoroughly. Some interesting applications of the formalism in analytic number theory are also presented. The error term as dictated by the Riemann hypothesis also follows naturally thus leading to an indirect proof of the hypothesis.

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1. Introduction

We present a simple proof of the Prime Number Theorem (PNT) [1] on the basis of a more refined treatment of the elementary concept of limit in the ordinary analysis. This work is in continuation of our current studies on the formulation of a scale invariant analysis that aims at developing a coherent framework for analysis on the real line R as well as Cantor like fractal subsets of R [5, 6]. The present proof is a more simplified version of the proof presented recently [7]. By Cantor sets we mean any compact, perfect, no-where dense subset of R. Although the results presented in this work should be valid for any such general Cantor set, we, for the sake of simplicity, restrict ourselves to a class of homogeneous Cantor sets defined as the limit set of an IFS of the form

$$f_1(t) = \beta t, \ f_2(t) = \beta t + (1 - \beta) \tag{1}$$

where $0 < \beta < 1$ and $\alpha + 2\beta = 1$. Both the box and Hausdorff dimensions of the limit set are known to be $s = \frac{\log 2}{\log \beta^{-1}}$. The precise nature of effects coming from a choice of the Cantor set will be studied in a more detailed analysis elsewhere.

To state the PNT, let $\Pi(x) = \sum_{p<x} 1$, p being the pth prime, be the prime counting function, i.e., $\Pi(x)$ counts the number of primes less than a given real number x. Then the PNT states that $\Pi(x)/(\frac{x}{\log x}) = 1$, as $x \to \infty$. The problem of determining the precise form of the correction term of this asymptotic formula is still unsettled. According to the Riemann hypothesis (RH), $\Pi(x)/(\frac{x}{\log x}) = 1 + O(x^{-\frac{1}{2}+\epsilon})$, as $x \to \infty$, for any $\epsilon > 0$. Although all the experimental searches on primes are known to agree with this RH correction, no analytical proof is still available in the literature [1]. In our approach, we study $\Pi(x^{-1})$, when $x \to 0^+$, so that the correction term, in consonance with the RH, has the form $\Pi(x^{-1}) \times x\log x^{-1} = 1 + O(x^{\frac{1}{2}-\epsilon})$, as $x \to 0^+$, for any $\epsilon > 0$. It is heartening to note that the present approach appears to produce the above RH correction term *naturally*, thus leading to a proof of the celebrated Riemann Hypothesis, though in an indirect way. The classical Hadamard- de la Vallée Poussin proof of the PNT vis-á-vis the present approach will be examined separately.

2. Scale Invariant formalism:

The formulation of a *scale invariant analysis* was motivated by our effort in justifying the construction of the so-called *nonsmooth* solutions [2] of the simplest scale invariant Cauchy problem

$$t\frac{dX}{dt} = X, \ X(1) = 1 \tag{2}$$

in a rigorous manner. It is clear that the framework of classical analysis, because of Picard's uniqueness theorem, can not rigorously accommodate such solutions, except possibly only in an approximate sense. To bypass the obstacle, it became imperative to look for a non-archimedean extension of the classical setting, thus allowing for existence of

nontrivial *infinitesimals* (and hence, by inversion, infinities). Robinson's original models of nonstandard analysis (or any minor variations of the same) appeared to be unsatisfactory, because (i) infinitesimals here are *infinitesimals* even in "values", (ii) the value of an infinitesimal is the usual Euclidean value and (iii) these are *new extraneous elements* in R (a comparison with Nelson's approach [4] with the present one will be considered elsewhere). Although, the nonstandard **R** *is* non-archimedean, but still an infinitesimal behaves more in a "real number like" manner; that is to say, in essence, it fails to have an *identity*, except for its infinitesimal Euclidean value. Such nonstandard infinitesimals are known to generate proofs of harder theorems of mathematical analysis in a more intuitively appealling manner (see, for example, [3]). Further, any new theorem proved in the nonstandard approach is expected to have a classical analysis proof, though, may be, using lengthier arguments. Justifying a higher derivative discontinuous (nonsmooth) solution of (2), therefore, appeared to be difficult even in the conventional nonstandard analysis.

To counter this problem, we contemplated developing a *novel* non-archimedean extension of R by completing the rational number field Q under a *novel* ultrametric which treats arbitrarily small and large rational numbers *seperately* from finite, *moderately large* rational (and real) numbers [7]. The ultrametric reduces to the usual Euclidean value for finite real numbers, but, nevertheless, leads to a *new* definition (realization) of scale invariant infinitesimals in the present context. An important feature of this formalism is that we make use of traditional $\epsilon - \delta$ techniques of classical analysis, but applied instead on a *deformed real number system* $\mathcal{R}$, when the deformation is induced by the ultrametric valuation of scale invariant relative infinitesimals (Definition 1,2). The straightforward derivation of the PNT in $\mathcal{R}$ clearly reveals the strength of this formalism, even at this relatively initial stage of its development.

We introduce scale invariant infinitesimals via a more refined evaluation of the limit $x \to 0^+$ in R. Notice that as $x \to 0^+$, there exists $\delta > 0$ such that $0 < \delta < x$ and one usually identifies zero (0) with the closed interval $I_\delta = [-\delta, \delta]$ at the *scale* (i.e. accuracy level in a computation) δ. Ordinarily, I_δ is a connected line segment, which shrinks to the singleton $\{0\}$ as $\delta \to 0^+$, so as to reproduce the infinitely accurate, *exactly determinable* ordinary real numbers. This also tells that in the usual (classical) sense there is no room for a scale when one talks about a limit of the form $x \to 0$.

Let us now present yet another *nontrivial mode* realizing the limiting motion $x \to 0^+$. This mode is nonlinear, as it is defined via an *inversion law*, rather than simply by linear translations, that is available *uniquely* to a real variable to undergo changes in the standard analysis on R. In the presence of *infinitesimals*, the present nonlinear mode is shown to gain significance. In the following we give a definition and also present a nontrivial construction for an *explicitly defined infinitesimal*, without requiring the set up of Robinson's nonstandard analysis. As remarked already, our formalism offers a new, independent realization of inifinitesimals residing originally in an ultrametric space, but, nevertheless, inducing nontrivial influences in the form of an *real valued "infinitesimal correction"* of the form $\delta(x) \propto x \log x^{-1}$ to an arbitrarily small real variable x approaching 0, thus making the existence of infinitesimals analytically (and dynamically as well) more *meaningful and significant*. As a consequence, a real number x, as it were, is raised to

deformed values $X_{\pm} \in \mathcal{R}$ (Sec. 3), because of a possible nontrivial motion close to 0. This construction also reveals the exact sense how a scale might become relevant in the present scale invariant formalism.

Definition 1. *[5, 6, 7, 8, 9] Let $x \in I = [-1, 1] \subset R$ and x be arbitrarily small, i.e., $x \neq 0$, but, nevertheless, $x \to 0^+$. Then there exists $\delta > 0$ and a set of (positive) relative infinitesimals $\tilde{x}$ relative to the scale δ satisfying $0 < \tilde{x} < \delta < x$ and the inversion rule $\tilde{x}/\delta \propto \delta/x$. The associated scale invariant infinitesimals are defined by $\tilde{X} = \lim_{\delta \to 0^+} \tilde{x}/\delta$.*

Example 1. *Let $x_n = \epsilon^{n(1-l)}$, $0 < l < 1$, $0 < \epsilon < 1$. Then scale invariant relative infinitesimals are $\tilde{X}_{n\lambda} = \lambda \epsilon^{nl}$, $0 < \lambda < 1$, when $\delta = \epsilon^n$, for a sufficiently large n, is chosen as a scale. Analogously, for a continuous variable x approaching 0^+, say, and considered as a scale, a class of the relative infinitesimals are represented as $\tilde{x} = x^{1+l}(1 + o(x))$, $0 < l < 1$, so that the corresponding scale invariant infinitesimals are defined by the asymptotic formula $\tilde{X} = \lambda x^l + o(x^m)$, $m > l$. Notice that a scale invariant infinitesimal goes to zero at a smaller (ultrametric) rate l: $\tilde{X} = \lambda x^l \Rightarrow \frac{d \log \tilde{X}}{d \log x} = l$.*

Remark 1. In the limit $\delta \to 0$, the set of relative infinitesimals apparently reduces to the singleton $\{0\}$. However, the set of scale invariant infinitesimals is nevertheless nontrivial, in the sense that these will have nontrivial influences on the real number system. To uncover the nature of such influences, the set of infinitesimals, denoted $\mathbf{0}$, is now awarded with a *natural* weight (absolute value) with nontrivial effects on the number theory and other areas of analysis. The set $\mathbf{0}$ has the *asymptotic representation* $\mathbf{0} = \{0, \delta \tilde{X}\}$, as $\delta \to 0^+$. For definiteness, the ordinary zero (0) is called the *stiff* zero, when the nontrivial infinitesimals are called *soft* zeros. The ordinary real line R is then extended over $\mathbf{R} = \{\tilde{\mathbf{x}} : \tilde{\mathbf{x}} = x + \mathbf{0}, \ x \in R\}$, which as a field extension, and as a consequence of the Frobenius theorem, must be an infinite dimensional nonarchimedean space. To re-emphasize, our aim here is to show nontriviality of soft zeros in obtaining an easy proof of the PNT.

Definition 2. *[5, 6, 7, 8, 9] An absolute value $v : \mathbf{0} \to I^+ = [0, 1]$*

$$v(\tilde{x}) \equiv ||\tilde{x}|| := \lim_{\delta \to 0^+} \log_{\delta^{-1}} \tilde{X}^{-1}, \ \tilde{x} = \delta \tilde{X} \ \in \mathbf{0} \tag{3}$$

together with $v(0) = 0$ is assigned to the set of infinitesimals $\mathbf{0}$. Infinitesimals weighted with above absolute value are called valued *infinitesimals.*

Let us list a few more interesting features of scale invariant infinitesimals and the associated absolute value [7, 8, 9].

Remark 2:

1. As remarked already, the set of infinitesimals $\mathbf{0} = \{0\}$ when $\epsilon \to 0$ classically. However, the corresponding asymptotic expressions for the scale free (invariant) infinitesimals are nontrivial, in the sense that the associated valuations (Definition 2) can be shown to exist as *finite real numbers*. Below we give a definite construction indicating the exact sense how relative infinitesimals and associated values arise in a limiting problem.

Fix a value of $\delta = \delta_0$ and let $C_{\delta_0} \subset [0, \delta_0] \equiv I_{\delta_0}^+$ be a Cantor set defined by an IFS of the form

$$f_1(x) = \lambda x, \ \ f_2(x) = \lambda x - (\lambda/\delta_0 - 1)\delta_0 \tag{4}$$

where $\lambda = \beta\delta_0$, $0 < \beta < 1$ and $\alpha + 2\beta = 1$. Thus, at the first iteration an open interval O_{11} of size $\alpha\delta_0$ is removed from the interval $I_{\delta_0}^+$, at the second iteration 2 open intervals O_{21} and O_{22} each of size $\alpha\delta_0(\beta)$ are removed and so on, so that a family of gaps O_{ij} of sizes $\alpha\delta_0(\beta)^{i-1}$, $j = 1, \ldots, 2^{i-1}$ are removed in subsequent iterations from each of the closed subintervals I_{ij}, $j = 1, 2, \ldots, 2^i$ of $I_{\delta_0}^+$. Consequently, $C_{\delta_0} = I_{\delta_0}^+ - \underset{i}{\cup} O_{ij} = \underset{i}{\cap}\underset{j}{\cup} I_{ij}$. Notice that the total length removed is $\sum \alpha\delta_0(2\beta)^{i-1} = \delta_0$, so that the linear Lebesgue measure $m(C_{\delta_0}) = 0$.

Next, consider $\tilde{I}_N = [0, \beta^N]$ and let $N = n + r$ and $N \to \infty$ as $n \to \infty$ for a fixed $r \geq 0$. Choose the scale $\delta = \alpha\beta^n\delta_0$ and define $\tilde{x}_r \in [0, \alpha\beta^N\delta_0]$ a (positive) *relative infinitesimal* (relative to the scale δ) provided it also satisfies the *inversion* rule $\tilde{x}/\delta = \lambda\delta/x$ (c.f. Definition 1), for a real constant $\lambda(\delta)$ ($0 \ll \lambda \leq 1$). For each choice of x and δ, we have a unique $\tilde{x}$ for a given $\lambda \in (0, 1)$. Consequently, by varying λ in an open subinterval of (0,1), we get an open interval of relative infinitesimals in the interval $(0, \delta)$, all of which are related to x by the inversion formula. In the limit $\delta \to 0$, relative infinitesimals $\tilde{x}_r$, of course, vanish identically. However, the corresponding scale invariant infinitesimals $\tilde{X}_r = \tilde{x}_r/\delta$, $\delta \to 0$ are, nevertheless, nontrivial and weighted with new *scale invariant absolute values* via Definition 2.

The set of infinitesimals are uncountable, and as shown below the above norm satisfies the stronger triangle inequality $v(x + y) \leq \max\{v(x), v(y)\}$. Accordingly, the zero set $\mathbf{0} = \{0, \pm\delta\tilde{X}_r | \ \tilde{X}_r \in (0, \beta^r), \ r = 0, 1, 2, \ldots, \ \delta \to 0^+\}$ may be said to acquire *dynamically* the structure of a Cantor like ultrametric space, for each $\beta \in (0, 1/2)$ (so as to satisfy the open set condition [7]). The set $\mathbf{0}$ indeed is realized as a set of nested circles S_r : $\{\tilde{x} | \ v(\tilde{x}_r) = \alpha_r\}$, in the ultrametric norm, when we order, without any loss of generality, $\alpha_0 > \alpha_1 > \ldots$. The ordinary 0 of R is replaced by this set of scale free infinitesimals $0 \to \bar{\mathbf{0}} = \mathbf{0}/\sim = \{0, \cup S_r\}$; $\bar{\mathbf{0}}$ being the equivalence class under the equivalence relation $\sim$, where $x \sim y$ means $v(x) = v(y)$. The existence of $\tilde{x}$ could also be concieved dynamically as a computational model [5, 6, 7], in which a number, for instance, 0 is identified as an interval $[-\delta, \delta]$ at an accuracy level determined by $\delta = \beta^n$.

2. The concept of infinitesimals and the associated absolute value considered here become significant only in a limiting problem (or process), which is reflected in the explicit presence of "$\lim_{\epsilon\to 0}$" in the relevant definitions. Recall that for the continuous real valued function $f(x) = x$, the statement $\lim_{x\to 0} x = 0$, means that $x \to 0$ essentially *is* $x = 0$. This may be considered to be a *passive* evaluation (interpretation) of limit. The present approach is *active (dynamic)*, in the sense that it offers not only a more refined evaluation of the limit, but also provides a clue how one may induce new (nonlinear) structures (ingredients) in the limiting (asymptotic) process. The inversion rule (Definition 1) is one such *nonlinear structure* which may act nontrivially as one investigates more carefully the *motion* of a real variable x (and hence of the associated scale $\delta < x$) as it goes to 0 more and more accurately. Notice that at any "instant", elements defined by inequalities $0 < \tilde{x} < \delta < x$ in a limiting process, are well defined; relative infinitesimals are *meaningful* only in that *dynamic* sense (classically, these are all zero, as x itself *is* zero). Scale

5

invariant infinitesimals $\tilde{X}$, however, may or may not be zero classically. $\tilde{X} = \mu \ (\neq 0)$, a constant, for instance, is nonzero even when x and ϵ go to zero. On the other hand, $\tilde{X} = \delta^\alpha$, $0 < \alpha < 1$, of course, vanish classically, but as shown below, are nontrivial in the present formalism. As a consequence, relative (and scale invariant) infinitesimals may said to *exist* even as real numbers in this dynamic sense. The acompaning metric $|| \cdot ||$, however, is an ultrametric.

3. However, a genuine (nontrivial) scale free infinitesimal $\tilde{X}$ can not be a constant. Let $\tilde{x}_0 = \mu \delta$, $0 < \mu < 1$, μ being a constant. Then $v(\tilde{x}_0) = \lim_{\delta \to 0} \log_\epsilon \mu = 0$, so that $\tilde{x}_0$ is essentially the trivial infinitesimal 0 (more precisely, such a relative infinitesimal belongs to the equivalence class of 0).

4. The scale free infinitesimals of the form $\tilde{X}_m \approx \delta^{\alpha_m} + o(\delta^\beta)$, $\beta > \alpha$ go to 0 at a slower rate compared to the linear motion of the scale δ. The associated nontrivial absolute value $v(\tilde{x}_m) = \alpha_m$ essentially quantifies this *decelerated* motion.

Theorem 1. *v has following properties.*
(i) v is an ultrametric, and hence $\mathbf{0}$ equipped with v is an ultrametric space.
(ii) v is a locally constant Cantor function.

Proof. (i) (a) v is well defined. Indeed, the open set $\mathbf{0}$ (in the usual topology, for each fixed δ) is written as a countable union of disjoint open intervals $I_{\delta i}$ of relative infinitesimals $\mathbf{0} = \bigcup I_{\delta i}$. Let $v(\tilde{x}_i) = \alpha_i$, a constant for all $\tilde{x}_i(= \lambda \delta \delta^{\alpha_i}) \in \bar{I}_{\delta i}$, the closure of $I_{\delta i}$. Thus v exists and well defined.

(b) Let $0 < \tilde{x}_2 < \tilde{x}_1 < \tilde{x}_1 + \tilde{x}_2 < \delta$ be two relative infinitesimals. We have, $0 < \tilde{X}_2 < \tilde{X}_1 < \tilde{X}_1 + \tilde{X}_2 < 1$ and $v(\tilde{x}_2) > v(\tilde{x}_1) > v(\tilde{x}_1 + \tilde{x}_2)$, thus proving the strong triangle inequality $v(\tilde{x}_1 + \tilde{x}_2) \leqq \max\{v(\tilde{x}_1), v(\tilde{x}_2)\}$.

Next, given $0 < \tilde{x} < \delta$, there exist a constant $0 < \sigma(\delta) < 1$ and $a : \mathbf{0} \to R$, such that $\tilde{X} = \lambda \delta^{v(\tilde{x})}$ and $v(\tilde{x}) = \sigma^{a(\tilde{x})}$. Accordingly, $a(\tilde{x})$ is a discrete valuation satisfying (i) $a(\tilde{x}_1 \tilde{x}_2) = a(\tilde{x}_1) + a(\tilde{x}_2)$, (ii) $a(\tilde{x}_1 + \tilde{x}_2) \geq \min\{a(\tilde{x}_1), a(\tilde{x}_2)\}$. As a result, $v(\tilde{x}_1 \tilde{x}_2) = v(\tilde{x}_1) v(\tilde{x}_2)$. Hence, $\{\mathbf{0}, v\}$ is an ultrametric space.

(ii) Let $\bar{\mathbf{0}} = (\cup \bar{I}_{\delta i}) \bigcup (\cup J_k)$, the closure of $\mathbf{0}$. The open intervals J_k are gaps between two consecutive closed intervals $\bar{I}_{\delta i}$. J_k's actually contain new points those arise as the limit points of sequences of the end points of the open intervals $I_{\delta i}$. Clearly, $\bar{\mathbf{0}}$ is connected in usual topology. However, in the ultrametric topology, both $I_{\delta i}$ and J_k are clopen sets and $\bar{\mathbf{0}}$ is totally disconnected. Since, it is bounded and also is perfect $\mathbf{0}$ is equivalent to an ultrametric Cantor set.

Now, the local constancy of v in the ultrametric $\bar{\mathbf{0}}$ follows from the definition:

$$\frac{dv(\tilde{x})}{dx} = \lim_{\delta \to 0^+} \frac{d}{dx} \left(\frac{\log x}{\log \delta} + 1 \right) = 0 \tag{5}$$

The vanishing derivative above arises from a logarithmic divergence arising from the nontrivial finer scales. This is unlike the ordinary analysis, when one interprets $\bar{\mathbf{0}}$ as a connected subset of R, thereby forcing v to vanish uniquely, so as to recover the usual structure of R. The above vanishing derivative can be interpreted nontrivially as a LCF when $x \in R$ is supposed to belong to a Cantor subset of I [8].

Eq(5) also reveals the *reparaterization invariance* of a locally constant valuation $v(x)$. As a consequence, v may be a function of any reparametrized monotonic variable $\tilde{x} = \tilde{x}(x)$ with $\tilde{x}'(x) > 0$, instead being simply a function of the original real variable x.

Remark 2. The differentiability in the metric space $\mathbf{0}$ is defined as usual : $f : \mathbf{0} \to R$ is differentiable at $x = a \in \mathbf{0}$ with derivative $f'(a)$ if $\lim_{x \to a} \frac{|f(x) - f(a)|}{||x - a||} = f'(a)$, when $x \to a$ means $||x - a|| \to 0^+$. In the above derivation, however, derivative is evaluated in usual metric which is equivalent to the natural ultrametric of the Cantor set inhabiting x. This also tells that differentiability in a Cantor set is well defined even in natural metric when increments in such a set is defined by inversions rather than by linear shifts [5, 6].

Now to construct a general class of locally constant functions in the ultrametric space, let us proceed as in (ia) above, with the supposition that the constants α_i's are arranged in ascending order. Thus, $v(\tilde{x}_i) = \alpha_i$, $\alpha_i \leq \alpha_j \Leftrightarrow i \leq j$ for all $\tilde{x}_i \in I_i$ (we drop the suffix δ for simplicity). Clearly, (5) holds over for all I_i. On the other hand, for an $\tilde{x} \in J_k$, where J_k separates two consecutive I_i and I_{i+1}, say, so that $\tilde{x}_i < \tilde{x} < \tilde{x}_{i+1}$, where $\tilde{x}_i$ is the right end point of I_i and $\tilde{x}_{i+1}$ is the left endpoint of I_{i+1}, we have $v(\tilde{x}_{i+1}) - v(\tilde{x}_i) = (\alpha_{i+1} - \alpha_i)$. Because of ultrametricity, one can always choose $\alpha_i = \beta_{ij_i} \sigma(i)^s$, for $\beta_{ij_i} > 0$ ascending and $\sigma(i) \to 0$ as $i \to \infty$ and $j_i = 0, 1, 2, \ldots k(i)$ for some i dependent constant $k(i)$ (c.f. 2nd para of (1b)). Consequently, $v(\tilde{x}_{i+1}) - v(\tilde{x}_i) = (\beta_{(i+1)j_{i+1}} - \beta_{ij_i})\sigma(i)^s$. It follows that the sequence $v(\tilde{x}_{i+1})$ is increasing and $v(\tilde{x}_i)$ is decreasing. Thus, $v(\tilde{x}) := \lim v(\tilde{x}_i)$ as $i \to \infty$. Hence, $v : \mathbf{0} \to I^+$ is indeed a Cantor function.

Conversely, given a Cantor function $\phi(x)$, $x \in I^+$, one can define a class of infinitesimals $\tilde{x} \approx \delta\delta^{\phi(\tilde{x}/\delta)}$ belonging to the extended set $\mathbf{0}$ for $\delta \to 0^+$. This completes the proof. ■

Remark 3. To re-emphasise, the valuation v can be considered to quantify the degree of *decelerated* motion as the real variable $x \to 0$ because of obstructions offered by nontrivial scale invariant infinitesimals in the ultrametric Cantor set of $\mathbf{0}$. The usual Euclidean norm is a measure of a finite real number because of its position relative to 0.

Definition 3. *Besides the usual Euclidean value, a real variable $x \neq 0$, but $x \to 0^+$ gets a deformed ultrametric value given by $v(x) := \lim_{\delta \to 0^+} \log_{\delta^{-1}}(x/\delta)$. Then $v(x) = v(\tilde{x})$.*

Proof. Because of inversion rule, $x/\delta = \lambda(\delta/\tilde{x})$, $0 < \lambda < 1$, and hence $v(x) = v(\tilde{x})$ since $\lim \log_{\delta^{-1}} \lambda^{-1} = 0$. ■

To proceed further, let us formulate some basic features of v revealing its nature of variability.

Lemma 1. *Let $0 < |x| < |x'|$ be two arbitrarily small real variables and δ be a scale such that $0 < \delta < |x - x'| < |x| < |x'|$. Then $v(x') = v(x)$.*

Proof. From definition 3, $v(x - x') < v(x) < v(x')$. But $x' = x + (x' - x)$. So by ultrametric inequality, $v(x') \leq \max\{v(x), v(x' - x)\} \leq v(x)$. ■

Lemma 2. *Let $0 < |x| < |x'|$ be two arbitrarily small real variables and δ and δ' be two scales such that $0 < \delta < |x| < \delta' < |x'|$. The corresponding scale invariant infinitesimals*

are $\tilde{X}$ and $\tilde{X}'$ with associated valuations $v(x)$ and $v(x')$. Then $v(x') = (\alpha/s)v(x)$, where $\alpha = \lim \log_{\tilde{X}} \tilde{X}'$, determines the gap size between $\tilde{X}$ and $\tilde{X}'$ and $s = \lim \frac{\log \delta'}{\log \delta}$ is the Hausdorff dimension of the Cantor set of infinitesimals as x, $x' \to 0$.

Proof. The proof follows from

$$\frac{v(x')}{v(x)} = \lim \frac{\log(x'/\delta')}{\log(x/\delta)} \times \lim \frac{\log \delta}{\log \delta'} \tag{6}$$

so that $\alpha = \lim \log_{x/\delta}(x'/\delta') = \lim \log_{\tilde{X}} \tilde{X}' \Rightarrow \tilde{X}' = X^\alpha(1 + O(\beta(x, x')))$, $\beta \to 0$ faster than the linear approach $x \to 0$. ∎

Corollary 1. *Let $0 < \delta < \delta' < x$ be two scales in association with an arbitrarily small real variable and $\tilde{X} = (x/\delta)^{-1}$ and $\tilde{X}' = (x/\delta')^{-1}$ be the corresponding scale invariant infinitesimals. Then $v(x') = (\alpha/s)v(x)$, where $\alpha = \lim \log_{\tilde{X}} \tilde{X}'$, determines the gap size between $\tilde{X}$ and $\tilde{X}'$ and $s = \lim \frac{\log \delta}{\log \delta'}$ is the Hausdorff dimension of the Cantor set of infinitesimals as x, $x' \to 0$.*

Definition 4. *A scale invariant jump is defined by the pure inversion $\tilde{X}' = X^{-1}$ with the scale invariant minimal jump size $\alpha = 1$. The jump size α thus runs over the set of natural numbers N.*

Remark 4. Lemma 1 characterizes the equivalence classes of infinitesimals with identical valuations. Subsequent lemma (and corollary) tells that the valuation v changes only when an infinitesimal from one equivalence class switches over to another class.

Summing up the above observations, we now state a general representation of relative infinitesimals and corresponding valuation.

Lemma 3. *[8] A relative infinitesimal $\tilde{x}$ relative to the scale δ has the dominant asymptotic form*

$$\tilde{x} = \delta \times \delta^l \times \delta^{\phi(\tilde{x}/\delta)}(1 + o(1)) \tag{7}$$

with associated valuation $v(\tilde{x}) = l + \phi(\tilde{x}/\delta)$, where $l \geq 0$ is a constant and ϕ is a nontrivial Cantor function.

Proof. The locally constant $v = v_0 + v_1$ solves $\frac{dv}{dx} = 0$ and so the above ansatz is the more general solution, with the trivial ultrametric valuation $v_0 = l$ and the nontrivial valuation $v_1 = \phi$. The representation for $\tilde{x}$ now follows from definition. ∎

Remark 5. As a real variable x and the associated scale $\delta < x$ approach 0, the corresponding infinitesimals $0 < \tilde{x} < \delta$ may live (in contrast to measure zero Cantor sets considered so far) in a positive measure Cantor set C_p, say. Such a possibility is already considered in [8] in relation to an interesting phenomenon of *growth of measure*. In such a case $v_0(\tilde{x}) = m(C_p) = l$, the Lebesgue measure of C_p. The nontrivial component v_1 then relates to the uncertainty (fatness) exponent of the positive measure 1-set. In this extended model (c.f. Sec. 4), the valuation quantifies the presence of *nontrivial* motion in a limiting process: v_0 gives the uniform scale invariant motion when v_1 arises from the associated nonuniformity stemming out from measure zero Cantor sets.

Definition 5. *The norm* $||.||$, *induced by v, in* $\mathbf{R}$ *is defined by* $||\mathbf{x}|| = x$, $0 \neq x \in R$, *but* $||x|| = v(x)$, *when* $x \in \mathbf{0}$.

Lemma 4. $\mathbf{R}$ *in the above norm is a locally compact ultrametric space. Moreover, the topology induced by this norm on punctured set $R - \{0\}$ is equivalent to the usual topology and the corresponding embedding $i : R \to \mathbf{R}$ is continuous.*

The proof is straightforward and so is not included.

Lemma 5. *The topology induced by $||.||$ in* $\mathbf{0}$ *is distinct from the usual one on the interval* $(0, \delta)$, $\delta \to 0^+$.

The proof follows from the observation that the a sequence of the form ϵ^{n-nl}, $0 < l < 1$ converges to 0 in the usual norm for all $l \neq 0$, but, nevertheless, converges to l in $||.||$, since $||\epsilon^{n-nl}|| = v(\epsilon^{n-nl}) = l$, when $\delta = \epsilon^n \to 0$ as $n \to \infty$ (c.f. Example 1 and Definition 2). We note, incidentally, that the sequence ϵ^n converges to 0 in both the norms (when the scale δ is identified with ϵ^n itself). □

For other possibilities see Example 2.

Geometrically, the singleton $\{0\}$ is replaced by an ultrametric Cantor set C and supposed to be attached 'vertically' at 0 in relation to the 'horizontal' real line. The valuation of infinitesimals living in $\mathbf{0}$ is associated to the valuation of the underlying Cantor set, and depends on the problem at hand. The class of Cantor sets that may be attached at 0 is rather huge, leading to the possibility of different limiting behaviours for a given sequence because of different choices of scale invariant infinitesimals $\tilde{X}_i$ living in the Cantor set C_i.

Example 2. *Consider the sequence $x_n = 2^{-n}$. The standard limit is 0. (i) Choose $\delta = p^{-n}$ as a scale for a prime $p > 2$ so that $0 < \delta < x_n$. Then the(positive) relative infinitesimals are of the form $\tilde{x}_n = \lambda(p/2)^{-n} p^{-n}$, $0 < \lambda < 1$ be a constant and $0 < \tilde{x}_n < \delta < x_n$. Then $||x_n|| = \lim_{\delta \to 0^+} \log_{\delta^{-1}} \tilde{X}_n^{-1} = 1 - \frac{\log 2}{\log p}$. Notice that infinitesimals $\tilde{x}_n$ belong to a Cantor set of Hausdorff dimension $\frac{\log 2}{\log p}$. Thus depending on the nature/choice of Cantor set containing infinitesimals, the limit might be different.*

Another important implication of the above observation is considered in the following sections.

To summarize, the ordinary real number system R is extended to a nonarchimedean space $\mathbf{R}$ with nontrivial valued infinitesimals. More details of the structure of $\mathbf{R}$ is available in Sec.5 and in [7]. These valued infinitesimals now, in turn, leave an imprint on the real number set so as to *deform* the original set [8]. As become evident the deformed system will be archimedean with the usual topology.

3. Deformed Real Number System

An ordinary real x is extended over to the fattened variables $X_\pm = x\mathcal{X}_\pm$, $\mathcal{X}_\pm = x^{\mp v(\tilde{x}/x)}$ (so that $X_+ > x$ and $X_- < x$). The fattened variables $X_\pm$ live in a space $\mathcal{R}$ called the *deformed real number set*. Clearly, $R \subset \mathcal{R}$, since $v(0) = 0$. The deformation is induced

by scale invariant infinitesimals $\tilde{X}$ living in the ultrametric space $\mathbf{0}/\{0\}$. Recall that, scale invariant elements of the extended ultrametric space $\mathbf{R}$ in an infinitesimally small neighbourhood of x belongs to an ultrametric Cantor set $\mathbf{C}$ (homeomorphically equivalent to a Z_p) and undergo changes by *inversions* of the form $\tilde{X}_\pm = \tilde{x}_\pm/x \to \tilde{X}'_\pm = \tilde{X}_\pm^{\pm e^h}$, $h \in \mathbf{0}$ [8]. Such ultrametric infinitesimals and the associated inversions and related properties in $\mathbf{R}$, in turn, leave an *imprint* on the deformed set $\mathcal{R}$, so that the real valued deformation factors $\mathcal{X}_\pm$ not only live in an associated Cantor set (in usual topology) $\mathcal{C} \subset R$, in the *deformed neighbourhood* of 1, but also can be thought to undergo changes by inversions of the form $\mathcal{X}_\pm \to \mathcal{X}_\pm' = \mathcal{X}_\pm^{\pm e^h}$ where h now is a *real parameter*. Notice that there exists a natural homeomorphism between $\mathbf{C}$ and $\mathcal{C} \subset \mathcal{R}$. As a consequence, $\mathcal{R}$ locally has the structure of a positive measure Cantor set. Notice that given a real number x and a scale δ, there exists a continuous mapping $f : R \to R$ such that $x = \delta f(\delta)$. In the deformed system, this mapping f is induced by ultrametric infinitesimals in the form $f(\delta) = \delta^{-v(\tilde{x})}$.

Now, let us recall that differential shift increments in R are defined by $x \to x' = x+h$ so that $dx = x' - x = h$, $h \to 0$. In the present case, the inversion induced *jump increments* in $\mathcal{R}$, over and above the usual shift differential, are defined by $d_j x = \log\log_{\mathcal{X}} \mathcal{X}' = \log\log(\tilde{x}_0/\tilde{x})$ (we drop the suffix $\pm$ for simplicity), where $\mathcal{X}' > \mathcal{X} > 1$, say. Indeed, for each $\mathcal{X}$ which can be interpreted as a scale (>1), there exists a class of 'infinitesimals' (actually, 'infinities') $\mathcal{X}'$ lying in a connected interval so that $\log_{\mathcal{X}} \mathcal{X}'$ tends to a nonzero constant (>1, say) α as the real variable $x \to 0$. As a consequence, a line segment acquires a fractal like structure: a singleton $\{x\}$ of the real number system R, under the scale invariant action of the said infinitesimals, now gets *deformed and fractured* into a Cantor like set. The scale invariant extension of the line segment inhabiting the variable $\mathcal{X}$ is the Cantor set $\mathcal{C} \subset R$ and $\alpha = \log_{\tilde{X}} \tilde{X}'$ gives an estimate of the size of a jump connecting two points $\mathcal{X}$ and $\mathcal{X}'$ in $\mathcal{C}$ (c.f. Lemma 2). Clearly, the above estimate can also be written as $\log(\tilde{x}_0/\tilde{x})$, where a point $\tilde{x}_0$ of a Cantor set is replaced by a connected segment over which the real variable $\tilde{x}$ (with a slight abuse of notation, we are here using the same symbol which denoted infinitesimals in $\mathbf{0}$) is supposed to live in [5, 6].

In the light of the above remarks, the following two lemmas now succintly encode the various incremental modes in the deformed space $\mathcal{R}$.

Lemma 6. *The scale invariant deformed variables $\mathcal{X}$ living in a neighbourhood of 1 in the Cantor like subsets of $\mathcal{R}/R$ undergo transitions by inversions $\mathcal{X} \to \mathcal{X}^{-\alpha}$, where $\alpha > 0$ is a multiple of a natural number.*

Lemma 7. *Let $x \in R$ and $X \in \mathcal{R}$. The differential increments for $X \in \mathcal{R}$ are classified as: (i) $X' = X+h$ for linear shifts taking over R, (ii) $X' = e^{h'}X \Rightarrow \log X' = \log X + h'$, for infinitesimal scaling and (iii) $X' = X^{-e^{h''}} \Rightarrow \log\log X' = \log\log X^{-1} + h''$, for nonlinear inversions on $\mathcal{R}/R$. As a consequence, the linear and nonlinear increments are related as $h' = \log h$ and $h'' = \log\log h$, where h, h' and h'' are sufficiently small real variables.*

These two lemmas are almost self explanatory in view of the above remarks and Lemma 2. The infinitesimal scaling is an effect of non-zero infinitesimals belonging to a specific equivalence class, having a nonzero *constant* valuation $v(\tilde{x}) = \alpha$, since $X = x \cdot \delta^{v(\tilde{x})} = x \cdot \delta^\alpha$,

over a connected gap. Nontrivial inversion induced variations are *revealed* only under *double logarithmic scales* of an ordianry linear variable, when there is a transition from one gap to another (that is to say, between two points of the underlying Cantor set , for more details see [6, 8]).

Theorem 2. *The Cantor function $v(\tilde{x})$ solves (5) everywhere in I^+, thereby inducing a smoothening of the ordinary derivative discontinuity at a point $\tilde{x}_0$ of the underlying Cantor set C, say. The actual variability of v is reflected at a logarithmic scale $\log(\tilde{x}_0/\tilde{x})$ in the deformed space $\mathcal{R}$, i.e.,*

$$\log y \frac{dv}{d\log y} = -v, \ y = \tilde{x}_0/\tilde{x} \tag{8}$$

when the $\tilde{x}_0 \in C$ is replaced by an infinitesimal connected line segment in which the variable $\tilde{x}$ lives.

Proof. The Cantor function $v : I \to I$ is constant on the gaps of the Cantor set $\mathcal{C}$. At a point of the Cantor set, the variation of v is given by Lemma 2. It follows from the fattened variable representation that v solves (8) thus establishing the variability of v in the logarithmic variables, viz, $d\log v = -d\log\log(x_0/\tilde{x})$, in the neighbourhood of each x close to 0 when both x and 0 are supposed to be embedded in $\mathcal{R}$. ∎

The proof of the PNT is derived when the real variable x is assumed to live in $\mathcal{R}$. But before delving into this let us explore another interesting feature related to measure and cardinality of a set.

4. Invariance of measure and Cardinality

The usual "cut and delete" process realizing a Cantor set seems to give a misleading idea that the Lebesgue measure of a set, for instance, [0,1] becomes zero under recursive applications of a contraction mapping of the form, say, $f(x) = ax$, $0 < a < 1$. However, the full Lebesgue measure could actually be preserved at every level of iterations: $1 = a^n \times p^{ns}$ for a given $p > a^{-1}$, thus leading to a Cantor set of fractal dimension $s = \frac{\log a^{-1}}{\log p}$, that remains attached to the limit point, viz. 0, of the said contraction [9]. For a finite length l, the above balance is achieved as $l = (a^n l) \times p^{ns}$. Notice that the reduction of the original Lebesgue measure is compensated multiplicatively by the corrective scaling factor p^{ns}, associated with the limiting Cantor set , that is thought to be formed dynamically as the points in the original set [0,1] are scrambled (jumbled) together gradually under successive applications of the contraction mapping so as to degenerate finally into a totally disconnected Cantor set having the s dimensional Hausdorff measure identically equal to the Lebesgue measure of the original connected set. More interestingly, the cardinality of the original set is also preserved. Recall that according to the classical analysis, the cardinality of the connected set $(0, \delta)$, $\delta = a^n$ is the continuum c for each n, but the limit set $\{0\}$ has cardinality 0. This singular behaviour remains unexplained in the standard approach. However, in the present scale invariant framework, the singularity is removed by a dynamic realization of a Cantor set of cardinality c. Moreover this Cantor set need

not have to be considered to be a meager set, since because of the above realization of the full Lebesgue measure as the s- Hausdorff measure.

The failure of the classical analysis to appreciate the above two facts appears to stem from its very simplistic treatment of limit. A real variable δ may tend to 0 either continuously assuming all the values from the connected interval $(0, \delta)$, or by jumps from one length say, $\delta[0, 1]$, to the next smaller length $\delta^2[0, 1]$ and so on, via contractions. These two processes are more or less indistinguishable in the standard analysis, both yielding the unique limit 0.

However, in real world applications, there are various situations where a scope for a nontrivial interpretation naturally arise. Consider a bounded interval of real line containing 0 laid out as an elastic string. Suppose a ball is rolled so as to approach slowly toward 0, and finally coming to rest at 0. The limiting concept of classical analysis applies here quite well. The physical (Euclidean) distance of the ball from 0 vanishes continuously, giving a satisfactory perception of the actual physical problem.

But, if we now think that the elastic string itself is contracted successively (without dissection), even in an idealistic situation, when the entire string is ultimately contracted to the limiting zero size, then the usual analytical picture is insufficient! Not a single point of the string is lost; the whole mass of points of the string is ultimately squeezed together to form, as it were, a blob, rather than a singleton set. In the light of the present scale invariant approach, it now transpires that the blob actually would be a Cantor like set accommodating all the points of the elastic string of real line in a *scrambled* manner. The original (initial) length of the string would be transformed in the logarithmic scale to the Hausdorff measure of the Cantor set.

The present measure invariance appears to be complimentary to the standard interpretation of Hausdorff measure as the invariance of mass function (or content): Under an IFS, the total mass content 1 of the set [0,1] gets uniformly distributed with value $1/r^s$ (r being the scale factor of the IFS) into p number fragmented set so that $p \times (1/r^s) = 1$, so that the limiting Cantor set so formed has the Hausdorff dimension $\frac{\log p}{\log r}$, having the s-Hausdorff measure as the invariant mass content of the set. In the present scenario, the reduced size of the original set is recovered exponentially as the powers of the scale factor $1/p$ of the limiting Cantor set for a $p > a^{-1}$. For a $p < a^{-1}$, on the other hand, one could recast the above measure invariance as $a^{1/s} \times p = 1$, thus realizing the Cantor set with Hausdorff dimension $\tilde{s} = 1/s = \frac{\log p}{\log a^{-1}}$.

To explain the above invariance, we note that a length $l < 1$ is preserved multiplicatively by creating a length L at an orthogonal direction so that the (hyperbolic) inversion rule $lL = 1$ holds good. This rule is preserved even in the limit $l \to 0$ thus creating a space in the form a Cantor set in the orthogonal direction. In the following theorem we consider a homogeneous Cantor set with one gap and two bridges so that $p = 2$.

Theorem 3. *An arbitrarily small closed interval of length $l(\approx 0)$ under the repeated applications of a contraction $f(x) = ax$, $0 < a < 1$ degenerates into a limiting Cantor set C_l of Hausdorff dimension $s = \frac{\log 2}{\log a^{-1}}$, $0 < a < 1/2$ in the orthogonal direction (y axis) of infinitesimally small elements. The differential measure of the size of the Cantor set is given by $L^{-1} = l^{1/s}$ when $l \to 0$.*

Proof. Under each applications of contraction the reduction of the original length $l \approx 0$ is balanced multiplicatively in the logarithmic scales following the rule

$$l = a^{n \log l} \times p^{n \log L} \tag{9}$$

where $n = r n_r$ and both $n \to \infty$ and $n_r \to \infty$ as $r \to \infty$, so that in the limit $r \to \infty$ we have $L^{-1} = l^{1/s}$, when the limit Cantor set has the Box (Hausdorff) dimension $s = \lim_{n_r \to \infty} \frac{\log 2^{n_r}}{\log a^{-n_r}}$ and a is restricted to $0 < a < 1/2$ to satisfy the open set condition and $p = 2$. The limit $r \to \infty$ actually normalizes the arbitrarily small l to 1 i.e., $l^{1/r} \to 1$, so as to allow retrieval of the original measure l at an exponentially smaller level and that too in the logarithmic scale.

For a $p > a^{-1}$, we can choose a Cantor set with scale factor $0 < b < 1/2$ so that $p = a^{-\frac{\log b^{-1}}{\log 2}}$, so that the desired differential measure follows when a is replaced by b. ∎

Remark 6. An arbitrarily small real variable $x \to 0$ now has the deformed value $X = x e^{\frac{1}{s} \log(x/\tilde{x})} \in \mathcal{R}$. As a consequence, the precise extension of the ordinary real line (in a neighbourhood of 0) is given by $X - x = (v(\tilde{x})/s) x \log x^{-1}$, when we set $x/\tilde{x} = x^{-v(\tilde{x}(x))}$, where v is the ultrametric valuation for infinitesimals $\tilde{x} \in \mathbf{0}$. Now, infinitesimals $\tilde{x}$ live in the gaps of a Cantor set in $(0,\delta)$ and v when extended over the closure of $(0,\delta)$ is equivalent to the Cantor function, having constant values α_i on the connected subintervals of countable number of gaps. As a consequence, for infinitesimals from any such gaps we have $X - x = (\alpha_i/s) x \log x^{-1}$, and hence as the real variable x goes to zero ultimately (that is becomes $O(\delta)$ and smaller), X still remains nonzero because of logarithmic correction factor. This new sublinear effect [9] is at the heart of the proof of the prime number theorem that we sketch below.

5. Prime Number Theorem

We begin by recalling a few salient features of infinitesimals and the associated influences on R.

1. Infinitesimals are real numbers in a limiting sense that these are elements of $(0,\delta)$ satisfying the inversion rule which is valid even in the limit $\delta \to 0^+$.

2. Scale invariant infinitesimals $\tilde{X} = \tilde{x}/\delta$, $\delta \to 0$ can not be a constant, i.e. $|\tilde{X}| = \lambda_0 < 1$, for then the corresponding valuation would vanish $v(\tilde{x}) = \lim \log_\delta \lambda_0^{-1} = 0$, thereby reducing the set of infinitesimals essentially to the trivial infinitesimal 0.

3. The nontrivial form of scale invariant infinitesimals are $\tilde{X} = \lambda \delta^{v(\tilde{x})}$ (which is not a constant) and so depends on the choice of scale δ, when the limit in the definition of v is evaluated with a slight modification of the original definition

$$v(\tilde{x}) := \lim_{n \to \infty} \log_{\delta^{-n}} \tilde{X}^{-1}, \quad \tilde{x} = \delta^n \tilde{X} \in \mathbf{0} \tag{10}$$

The scale invariant infinitesimals are therefore also said to be scale invariant $\delta-$ infinitesimals [7]. The transition between two δ infinitesimals $\tilde{X}$ and $\tilde{X}'$ would be mediated by inversions (jumps) of the form $\tilde{X} \to \tilde{X}' = \tilde{X}^{-\alpha}$, where α takes values from {range}$v(\tilde{x}) = N$.

4. The valuation v defined on the set of infinitesimals is a nontrivial ultrametric. Since the real number system can not have any nontrivial metric other than the usual one, the set of $\delta-$ infinitesimals must be a *non-real* ultrametric space. Now, given the rational number set Q, there exists inequivalent p-adic local fields Q_p, for each prime (finite) p, together with $Q_\infty \equiv R$. Next, as a consequence of the Frobenius theorem any ultrametric extension of R must be infinite dimensional. Hence, **R** is an infinite dimensional metrically complete ultrametric space which locally has the product structure $\mathbf{0} = \prod Q_p$ (c.f. Theorem 1) [7]. However, it is also well-known that ultrametric spaces has an *hierarchical* structure [10]. Each fibre Q_p is, therefore, called a branch or leaf of the ultrametric space and a corresponding p infinitesimal $\tilde{X}_p$ (say) living in the pth branch Q_p would transit to the immediate successor $\tilde{X}_q$ living in the Q_q branch for the prime q being the next prime exceeding p via inversion induced jumps.

5. As a real variable $x > 0$ approaches 0, because of continuity, it is supposed to vanish ultimately. However, because of the scale invariant infinitesimals, it is, however, replaced by the deformed $X = x + (v(\tilde{x})/s)x \log x^{-1}$, living in $\mathcal{R}$, which is exact for sufficiently small values of x. The first term x is the usual *linear* variable which goes to zero in the limit. The second is a nontrivial correction, which we investigate more in detail below. Notice that as the dominant term x goes to zero in the limit $x \to 0(\sim \delta)$, the *motion (variation)* in the sub-dominant correction term is *transmitted* to $v(\tilde{x})$, when the factor $x \log x^{-1} \sim O(\delta \log \delta^{-1})$ now acts as an *infinitesimal* scaling factor. To emphasize, this is a nontrivial real valued infinitesimal living in the archimedean real number system.

6. First, we recall that the ultrametric norm quantifies the degree of nontrivial *motion* experienced by a real variable x in presence of nontrivial scale invariant infinitesimals in the zero set $\mathbf{0} = \prod Q_p$. The uniform rate 1 of approach toward 0 of the original variable x gets flickered because of multiple interjections from scale invariant infinitesimals $\tilde{X}_p \in Z_p \subset Q_p$, the ring of p-adic integers.

7. As the variable x becomes vanishingly small (so that the first term of X gets subdominant compared to the second), its *linear* motion gets transferred to the scale invariant ultrametric variable $\tilde{X}_2$ living in $Z_2 \subset Q_2$. However, *the nonvanishing contributions to the deformed real variable X* from the scale invariant $\tilde{X}_2$ is received in the form of *real values* via the norm v, viz, $X = v(\tilde{x}_2)\delta \log \delta^{-1/s}$, where we neglect x and $\tilde{x}_2 = \delta 2^n \tilde{X}_2 \in Z_2$, $n \to \infty$.

8. Recall that v is a locally constant function, $dv/dx = 0$, but its variability would get revealed in a growing real variable x_2 in the neighbourhood of x: existence of $x_2(x)$ is assured by scale invariant infinitesimals. Recall $v(\tilde{x}_2) = \log_{\delta^{-n}} \tilde{X}_2^{-1}$ and $\tilde{X}_2 = \delta^{n\alpha \log(x/x_2)}$, as $n \to \infty$, so that $v(\tilde{x}_2) = \alpha \log(x/x_2)$, establishing that $dv/dx = 0$, but $ydv/dy = v$, $y = (x/x_2)$, since $xdx_2/dx = x_2$. Notice that as $x \to 0^+$, in the neighbourhood of each such $x \neq 0$, $x_2 \to x$, in such a manner that $x/x_2 = 1 + \eta_2$, where the scale invariant η_2 is a growing variable from 0 in (0,1). As a consequence, the deformed X would initially have the approximate form $X \approx (\alpha/s)\eta_2\delta \log \delta^{-1}$, which grows with η_2 until η_2 becomes of the order of $O(1)$, viz, $\eta_2 = 1 - \eta_3$; $\eta_3 \approx 0$ is another scale invariant variable living in (0,1), when the motion of η_2 is transferred to η_3 by inversion $O(1) \sim \eta_2 \to (1 - \eta_2)^{-1} = 1 + \eta_3$ so as to induce a slightly altered value of the deformed variable $X = (v(\tilde{x}_3)/s)\delta \log \delta^{-1} = (\alpha/s)(1 + \eta_3)\delta \log \delta^{-1}$, and the initially negligible η_3 grows continuously to $O(1)$ in the

interval (0,1). Notice that $\tilde{x}_3 \in Z_3$ and $x/x_3 = 1 + \eta_3$.

9. The above general structure will replicate each time the scale invariant infinitesimal $\tilde{X}_p$ living in Z_p switches over to an $\tilde{X}_q$ in Z_q, for the prime q immediate successor of p, thus leaving an imprint over the deformed real X in the form $X = (v(\tilde{x}_q)/s)\delta \log \delta^{-1} = (\alpha/s)(1 + 1 + \ldots + 1 + \eta_q)\delta \log \delta^{-1}$, where the sum of 1's is up to pth prime and η_q grows linearly from 0 up to 1 in (0,1). The scale invariant ultrametric infinitesimals $\tilde{X}_p$ and $\tilde{X}_q$ switches between them by inversions. Analogous switching between scale invariant deformed real variables $x_p = 1 - \eta_p$ and $x_q = 1 + \eta_q$ is also mediated by inversion $x_q = x_p^{-1}$.

10. As a consequence, as $x \to 0$ more and more *accurately*, the scale invariant components x_p of the deformed extension X continue to grow, as the corresponding scale invariant infinitesimals continue to evolve over various branches of Z_p's as $p \to \infty$, until the asymptotic formula for the prime counting function $\Pi(x^{-1}) = 1 + 1 + \ldots + 1 + \eta_q$ becomes sufficiently large so as to cancel exactly the corrected (real) variable (now acting as such as a scale factor) $\delta \log \delta^{-1}$. This already completes the proof of the PNT.

11. As an aside, let us now observe the following:

Proposition 1. *The prime counting function $\Pi(x)$ is a locally constant function.*

To prove this, let us first consider the step function

$$f(x) = \begin{cases} a, & 0 < x < p, \\ b, & x > p \end{cases} \tag{11}$$

with a finite discontinuity at $x = p$ in the usual sense. In the present scale invariant formalism with inversion mode for increments, we now show that f solves $x\frac{df}{dx} = 0$ every where, that is, even at $x = p$. As x increases toward p from the left linearly, the graph of f is a straight line parallel to the x-axis. In the left neighbourhood of p, $x = p - \eta = px_-$, $x_- = 1 - \eta/p$. Analogously, in the right neighbourhood, we have $x = p + \tilde{\eta} = px_+$, $x_+ = 1 + \tilde{\eta}/p$, so that $x_+ = x_-^{-1}$. Let us assume that η and $\tilde{\eta}$ are sufficiently small, so that the point set $\{p\}$ is identified with the closed interval $I_p = [1 - \eta/p, 1 + \tilde{\eta}/p]$, and so defines the accuracy level of a given computational problem. The interval I_p corresponds to an infinitesimally small neighbourhood of p. At the level of this infinitesimal scale, the function f is interpolated by the scale invariant formula

$$\tilde{f}(x) = \begin{cases} a, & 0 < x < px_-, \\ a + (b - a)\phi_p(x), & x \in pI_p, \\ b, & px_+ < x. \end{cases} \tag{12}$$

where $x = (p - \eta) + (\tilde{\eta} + \eta)\tilde{x}$, $0 \leq \tilde{x} \leq 1$. Clearly, $\tilde{f}(x) = f(x)$, in the limit η, $\tilde{\eta} \to 0$. Moreover, $x\frac{d\tilde{f}}{dx} = 0$ everywhere, including $x = p$, since the locally constant Cantor function $\phi_p(x)$ on I_p does. It follows, therefore, that as x approaches to p from left and arrives at a point of the form $x = px_-$, it switches smoothly to $x = px_+$ at the right of p by inversion $x_-^{-1} = x_+$. The associated value of the function f i.e. a, however, changes over to b by a cascade of smaller scale self similar smooth jumps as represented by the Cantor function $\phi_p(x)$. The cost of this smoothness, however, is the arbitrariness in the formalism that is introduced via the arbitrariness of the choice of the Cantor function.

15

The prime counting function $\Pi(x)$ is a step function in the neighbourhood of every prime. Hence the result. $\square$

As another interesting application of the present formalism, let us re-examine the rate of divergences of the well known harmonic series and the series of reciprocals of primes.

Let $H_n = \sum_1^n \frac{1}{i}$, and $P_n = \sum_1^{p_n} \frac{1}{p}$, be the nth partial sums of the harmonic series and that of the reciprocals of primes respectively. Using a continuous variable $x > 0$, we get two infinite ladder functions $H(x) = \sum_{i \leq x} \frac{1}{i}$ and $P(x) = \sum_{p \leq x} \frac{1}{p}$. It is well known that $H(x) \sim \log x$ and $P(x) \sim \log \log x$, for sufficiently large $x \to \infty$. A basic difference between two infinite step functions is that $H(x)$ grows with uniform step (gap) size 1, when that for $P(x)$ is *nonuniform* and gap size grows slowly with primes.

To study the variability of $H(x)$ as $x \to \infty$, let us notice trivially that $\frac{dH}{dx} = 0$, for $n - 1 < x < n$. But, in the neighbourhood of $x = n$ (say), $H(x - h) - H(x) = n^{-1}$, $h > 0$. As n approaches ∞ uniformly, the scale invariant real variable n^{-1} gets a deformation factor $\mathcal{X}_+ = (n^{-1})^{-l} \approx hx^l$ for an arbitrary $l > 0$ (h here stands for the infinitesimal scaling of Lemma 7 and l to the uniform jump rate, c.f. Remark 5), so that we have $\Delta H(x) \sim hx^{-1+l} \sim \Delta x^l$. But, since $l > 0$ may be arbitrarily small, $x^l \sim \log x$, as $x \to \infty$, thus leading to the desired rate $H(x) \sim \log x$.

For the series of primes, uniform inversion induced jumps now tell that, as $x \to \infty$, the ladder function $P(x)$ more and more behave as a Cantor function and hence asymptotically must vary as double logarithms $P(x) \sim \log \log x$ (Lemma 7, Theorem 3). This seemingly universal behaviour is likely to be present in any variations purely over primes and induced by inversion mediated smooth jumps.

Estimation of Euler and Martens constants γ and ρ respectively require more work, and is left for future.

6. Error term

Now, returning to the PNT, we note that the inversion mediated motion that is being treated here of the deformed scale invariant variable X because of the motion of the scale invariant infinitesimals over various branches of the ultrametric space $\mathbf{0}$ may be considered to be the *internal motion*, relative to the original *external* motion of the real variable x as it approaches 0 linearly. Further, *there are two modes of changes available to a scale invariant $\eta \lessgtr 1$: (i) local mode, when infinitesimals get contributions from infinitely large scales, defined by $\eta \to \eta' = (1+\tilde{\eta})^{-1}(< 1)$ and (ii) global mode, $\eta \to \eta' = \eta^{-1} = 1+\tilde{\eta}(> 1)$, where $\tilde{\eta}$ is another growing variable living in a branch immediate successor to that of η.* Next, we observe that in the scale invariant formalism 1 is replaced by $\mathbf{1}$, which asymptotically has the representation $\mathbf{1} = \tau(1 - \delta^{1/\tau})$, where $\tau = (1 + \epsilon\eta')$, ϵ being an appropriate scale factor and η' is a slowly varying growing variable in (0,1). Translating over in R, this equality must be interpreted as $x_- x_+ \approx 1$ close to 1 ($x_- \lessapprox 1$, $x_+ \gtrapprox 1$). We also notice that in the present formalism there arise various scales in connection with an arbitrarily small x: the primary scale $\delta_0 < x$, and associated secondary p-adic scales p^{-n}, so that the composite scale that becomes relevant in the definition of the nontrivial value v

is given as $\delta = \delta_0 p^{-n}$, $n \to \infty$. As the corresponding inverted ultrametric variable $\tilde{x}$ varies over local p-adic fields, thereby accumulating extra nontrivial additive corrections along the growing mode, there also arises another nontrivial scale, we call it as the "infinitesimal scaling factor", $\epsilon(\delta) = \delta \log \delta^{-1}$ leading to a realization of the PNT, as indicated above, via a cancellation of the two factors as $p \to \infty$.

As a consequence, the scale invariant extension of an arbitrarily small x (c.f. Remark 6) can be rewritten as

$$X - x = \epsilon(x)(x'/x)(1 - x^{v(x/\tilde{x})}) \tag{13}$$

where $\epsilon(x) = x \log x^{-1}$ ($\sim O(\delta \log \delta^{-1})$) acts as the nontrivial "infinitesimal" scaling factor, the factor $x'/x = 1 + \eta$ corresponds to a growing variable following the above growing mode, x' being a reparametrized variable in the neighbourhood of x and the remaining exponential term x^v is a function of the slower variable $\tau = \log(x/x') \approx \epsilon(x)(x'/x)$. Notice also that $v = 1/\tau = 1/(1 + \epsilon \eta')$, where η' is another very slowly growing variable (actually the prime counting function $\Pi(x^{-1})$).

Taking into consideration the above scaling effects the PNT now assumes the form

$$X \approx (1 - x^{\frac{1}{1+\epsilon\eta'}})(1 + 1 + \ldots + 1 + \eta_q)x \log x^{-1} = (1 - x^{\frac{1}{1+\epsilon\Pi(x^{-1})}})\Pi(x^{-1})\epsilon(x) \tag{14}$$

so that the final form of the deformed scale invariant X has the value $X = 1$, in the limit $x \to 0$ when the PNT with correction term has the form

$$\Pi(x^{-1})x \log x^{-1} = (1 - x^{\frac{1}{1+\epsilon\Pi(x^{-1})}}]^{-1} = 1 + O(x^{\frac{1}{2}-\epsilon'}) \tag{15}$$

for an $\epsilon' > 0$. As $x \to 0$, classically (i.e., becomes of $O(\delta)$ and less in a computational problem with accuracy δ), the scale invariant deformed variable X continues to evolve, following the above pattern, as the associated dynamical infinitesimal $\tilde{X}_p$ varies over various local fields Q_p, thus contributing successively a unity for every p. This process of *internal* evolution in the structure of the deformed scale invariant variable X continues until the prime counting function cancels the infinitesimal scaling factor ϵ, in both the growing as well as the local modes of eq(14), in a manner consistent with eq(15). Assuming $\Pi(x^{-1})x \log x^{-1} = 1 + \sigma$, the consistency means the equality $\sigma = x^{\frac{1}{2+\sigma}}$, for an arbitrarily small $\sigma(x)$. Clearly, eq(15) realizes precisely the error term dictated by the Riemann Hypothesis, and hence constitues an indirect proof of the same. One may interpret this error term as a reflection of the presence of randomness in the variations of scale invariant components of the form $\mathcal{X}$ of a deformed real variable in the neighbourhood of 1, as such a scale invariant variable lives in and varies over a Cantor set. However, this randomness is, of course, not a pure white noise, as it is evident from the nonzero $\epsilon' > 0$.

In the classical analysis $X = 0$, in the limit $x \to 0$. The counterintuitive growth of the value of X in the scale invariant formalism may be explained in the light of invariance and the associated growth of measure in Sec. 4. The exact real numbers of classical analysis with fixed (Euclidean/Archimedean) values are replaced in the scale invariant formalism by a collection of infinitesimally small neighbours (as, may be recalled, ordinary 0 is replaced by a Cantor set of soft zeros), with nontrivial effects on the value of a

real number, which, as shown above, is likely to become activated in a limiting process involving many infinitely large scales. Such a limiting process is abundantly available in complex systems, for instance, in many body systems in astronomy and astrophysics, in biological systems, in finance and similar others fields. The standard classical analytic results are, nevertheless, expected to be valid in a process (context) involving moderate scales ($\delta \sim O(1)$). To cite an example of growth of value, let us recall the variability of the *value* of money: money left in a locker at home over a few days or weeks keeps its value, but put in a bank or in (protected) share market, that is, when money is made to "flow", gains in value. The concept of limit in classical analysis is *passive*, that is, $x \to 0$ means $x = 0$ (recall that uniform (in the usual topology) motion is equivalent to "no" motion in R). The nonarchimedean valuation of soft zeros, however, make the uniform motion of $x \to 0$ nonuniform (as stated in Remark 5, this nonuniformity includes both uniform and nonuniform variations in ultrametric topology). As a consequence, the scale invariant extension of x, viz, $\mathcal{X} = X/x$ assumes a *directed evolutionary* character, thereby driving the limiting value of X not only to 1, but in the process also yields a novel proof of the PNT with correction term that follows from the Riemann Hypothesis. However, 1 in the scale invariant formalism is never exactly 1, but the fattened **1**, so that we now actually have $X = 1 \approx (1 - x^{\frac{1}{\tau_1}})\tau_1$, where τ_1 is a higher order O(1) slowly growing variable involving higher order prime counting function for the logarithmic variable $\log x^{-1}$, and so on.

7. Final Remarks

The proof of the prime number theorem and Riemann's hypothesis as well as the concept of novel invariance properties of measure and cardinality considered here are some what counterintuitive in the standard, conventional approach. However, these seem to follow quite naturally and elegantly from the so called internal motion of an inverted variable in an infinite dimensional ultrametric space accommodating dynamic infinitesimals. The ordinary singleton set of 0 is fractured into an ultrametric Cantor set, transitions between points of which are accomplished by inversion induced smooth jumps. Ordinary jump processes in probability and stochastic processes are nonsmooth and require a lot more technical details. The smooth jumps are equivalently interpreted as double logarithmic variations of an ordinary real variable. The associated deformed real number system $\mathcal{R}$ accordingly has the structure of a positive measure Cantor set, thus endowing the linear motion of $x \to 0$ in R with a nontrivial sublinear flow $X - x \approx x \log x^{-1}$, as $x \to 0$. Before closing, we note that the possible connection of a first order ODE of the form (2) with Riemann's hypothesis appears to have been first pointed out in Ref.[11] in the context of quntum mechanics.

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